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Finite Difference Approximations of Generalized Solutions

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Abstract. We consider finite difference schemes approximating the Dirichlet problem for the Poisson equation. We provide scales of error estimates in discrete Sobolev-like norms assuming that the generalized solution belongs to a nonnegative order Sobolev space.

1. Introduction. Recently, there have been many theoretical advances in constructing finite difference schemes approximating boundary value problems for partial differential equations with generalized solutions belonging to Sobolev spaces. For example, Lazarov [4] presents a finite difference approximation of the Dirichlet problem for the Poisson equation with a generalized solution belonging to the Sobolev space $W^{k,2}$ of integer order $k = 2, 3$ using the so-called Bramble-Hilbert lemma [1].

Unfortunately, the Bramble-Hilbert lemma is stated only for integer-order Sobolev spaces. Recently, Dupont and Scott [3] gave a constructive proof of this lemma using an averaged Taylor series and extended it to fractional-order Sobolev spaces.

In this paper a basic framework is given which allows the application of the finite difference method in order to approximate generalized solutions belonging to Sobolev spaces $W^{s,p}$, $0 \leqslant s \leqslant 4$, $1 < p < \infty$ (Theorems 1 and 3). Proofs are based on the Dupont-Scott approximation theorem.

We shall prove a discrete interpolation inequality (Lemma 2) which will enable us to derive several scales of error estimates (Theorems 2 and 4).

For simplicity, the analysis in this paper only deals with the Dirichlet problem for the Poisson equation in rectangular domains. Extensions to other elliptic boundary value problems in less special domains or to nonlinear problems are possible.

2. Preliminaries and Notations. Let $\mathscr{A}$ be an open rectangle in two-dimensional Euclidean space $\mathbf{R}^2$ and $1 < p < \infty$. Throughout the paper $W^{s,p}(\mathscr{A})$ is the Sobolev space of order $s \geqslant 0$ (cf. [8]) equipped with the Sobolev norm

$$\|u\|_{s,p,\mathscr{A}}^p = \sum_{k=0}^{s} |u|_{k,p,\mathscr{A}}^p$$

with

$$|u|_{k,p,\mathscr{A}}^p \doteq \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\mathscr{A})}^p,$$

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if s is integer, and

$$\|u\|_{s,p,\mathscr{A}}^p = \|u\|_{[s],p,\mathscr{A}}^p + |u|_{s,p,\mathscr{A}}^p,$$

if $s = [s] + \sigma$, with $[s]$ = integral part of s, $0 < \sigma < 1$ and

$$|u|_{s,p,\mathscr{A}}^p = \sum_{|\alpha|=[s]} \int_{\mathscr{A}} \int_{\mathscr{A}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{2+\sigma p}} \, dx \, dy.$$

$\mathbf{N}$ will stand for the set of nonnegative integers. $\mathbf{P}^l(\mathscr{A})$ will denote the set of polynomials in two variables of degree $\leqslant l$ over the set $\mathscr{A}$, for any $l \in \mathbf{N}$.

The next lemma is an easy consequence of the Dupont-Scott approximation theorem [3] (the case $\sigma = 1, p = 2$ follows from the Bramble-Hilbert lemma [1]).

LEMMA 1. *Suppose $s = l + \sigma$, where $0 < \sigma \leqslant 1$ and $l \in \mathbf{N}$. Let η be a bounded linear functional on $W^{s,p}(\mathscr{A})$ such that $\mathbf{P}^l(\mathscr{A}) \subset$ kernel(η). There exists a positive constant C (depending on $\mathscr{A}$, s, p) such that for any $u \in W^{s,p}(\mathscr{A})$*

$$|\eta(u)| \leqslant C|u|_{s,p,\mathscr{A}}.$$

Remark 1. Lemma 1 also follows from the Tartar lemma [2].

Remark 2. If $\eta(u) = 0$ for some polynomials of degree $> l$, then an analogous estimate is valid, containing only a part of the seminorm $|u|_{s,p,\mathscr{A}}$ (cf. Lazarov [4], $s \in N, p = 2$).

Let $\mathscr{D}'(\mathscr{O})$ denote the space of distributions on $\mathscr{O}$, for any open set $\mathscr{O} \subset R^2$. Define the differential operator Δ on $\mathscr{D}'(\mathscr{O})$ by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

Let us assume, for the sake of simplicity, that Ω is an open rectangle in R^2 with boundary $\partial\Omega$, and consider the Dirichlet problem

$$\Delta u = -f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

By changing variables, we may suppose, without loss of generality, that the rectangle is

$$\Omega = (0, \pi) \times (0, \pi).$$

Throughout the paper we assume that (1) has a unique generalized (distributional) solution in $W^{s,p}(\Omega)$, $0 \leqslant s \leqslant 4$, $1 < p < \infty$, satisfying (2) in the sense of trace theorems [5], [8].

3. Mollifiers. Consider the function

$$S_\nu(x) = \begin{cases} \left(\dfrac{\sin(x/2)}{x/2}\right)^\nu, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

with $\nu \in \mathbf{N}$. By the Paley-Wiener-Schwartz theorem [7] there exists a distribution θ_ν with compact support and with a Fourier transform equal to S_ν.

Remark 3. An easy argument shows that θ_0 is the Dirac distribution. For $\nu \geqslant 1$, θ_ν is a regular distribution. For example,

$$\theta_1(x) = \begin{cases} 1, & x \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ 0, & x \notin \left(-\frac{1}{2}, \frac{1}{2}\right), \end{cases}$$

$$\theta_2(x) = \begin{cases} 1 - |x|, & x \in (-1, 1), \\ 0, & x \notin (-1, 1), \end{cases}$$

$$\theta_3(x) = \begin{cases} \frac{1}{2}\left(x + \frac{3}{2}\right)^2, & x \in \left(-\frac{3}{2}, -\frac{1}{2}\right], \\ \frac{3}{4} - x^2, & x \in \left[-\frac{1}{2}, +\frac{1}{2}\right], \\ \frac{1}{2}\left(x - \frac{3}{2}\right)^2, & x \in \left[+\frac{1}{2}, +\frac{3}{2}\right), \\ 0, & x \notin \left(-\frac{3}{2}, +\frac{3}{2}\right). \end{cases}$$

Let $\nu = (\nu_1, \nu_2)$, $\nu_1, \nu_2 \in \mathbf{N}$, $x = (x_1, x_2) \in \mathbf{R}^2$, θ_ν the tensor product of distributions θ_{ν_1} and θ_{ν_2}, G_ν a distribution defined by

$$G_\nu(x) = \frac{1}{h^2}\theta_\nu\left(\frac{x}{h}\right), \qquad h > 0,$$

and $u \in \mathscr{D}'(R^2)$. The operator T_ν given by

$$T_\nu u = u * G_\nu$$

will be called mollifier.

Remark 4. Since G_ν is a distribution with compact support, the convolution $u * G_\nu$ is well defined.

For $h > 0$ and $\nu = (\nu_1, \nu_2)$, we set

$$\Omega_\nu = \left\{x = (x_1, x_2) \in \mathbf{R}^2 \colon h\nu_i/2 < x_i < \pi - h\nu_i/2, i = 1, 2\right\}.$$

Let $u \in \mathscr{D}'(\Omega)$ and $u^* \in \mathscr{D}'(R^2)$ be any extension of u. $T_\nu u$ will denote the restriction of $T_\nu u^*$ to Ω_ν.

Remark 5. Let us observe that $T_\nu u$ is well defined since it does not depend on u^*.

For simplicity, we shall write $T_{\nu_1\nu_2}$ instead of $T_{(\nu_1,\nu_2)}$.

4. Construction of Difference Schemes. Pick a nonnegative integer $N > 2$ and let $h = \pi/N$. We define the following grids

$$\mathbf{R}_h^2 = \left\{x = \left(x_1^{(i_1)}, x_2^{(i_2)}\right) \in \mathbf{R}^2 \colon x_j^{(i_j)} = i_j \cdot h,\ |i_j| < \infty, j = 1, 2\right\},$$

$$\omega_h = \Omega \cap \mathbf{R}_h^2, \qquad \gamma_h = \partial\Omega \cap \mathbf{R}_h^2,$$

$$\overline{\omega}_h = \omega_h \cup \gamma_h, \qquad \gamma_h^1 = \gamma_h \cap (\{0, \pi\} \times (0, \pi)),$$

$$\gamma_h^2 = \gamma_h \cap ((0, \pi) \times \{0, \pi\}),$$

$$\gamma_h^3 = \gamma_h \cap (\{0\} \times (0, \pi) \cup (0, \pi) \times \{0\}),$$

$$\omega_h^+ = \omega_h \cup \gamma_h^3.$$

For v, a function of discrete arguments, defined on $\mathbf{R}_h^2$, set

$$(\nabla_j v)(x) = \frac{v(x + e_j h) - v(x)}{h}, \qquad j = 1, 2,$$

$$(\overline{\nabla}_j v)(x) = \frac{v(x) - v(x - e_j h)}{h}, \qquad j = 1, 2,$$

with $e_1 = (1,0)$, $e_2 = (0,1)$, and define

$$\Delta_h v = \overline{\nabla}_1 \nabla_1 v + \overline{\nabla}_2 \nabla_2 v.$$

A function v of discrete arguments defined on ω_h (or on $\overline{\omega}_h$ and equal to zero on γ_h) is said to belong to $L^p(\omega_h)$, $1 < p < \infty$, if there exists a positive constant M, such that

$$\|v\|_{p,h} = \left\{ h^2 \sum_{x \in \omega_h} |v(x)|^p \right\}^{1/p} \leqslant M,$$

uniformly in h.

Remark 6. If v is defined on ω_h^+ (or on $\overline{\omega}_h$ and equal to zero on $\gamma_h \setminus \gamma_h^3$) the norm $\| \cdot \|_{p,h}$ is replaced by

$$|[v \|_{p,h} = \left\{ h^2 \sum_{x \in \omega_h^+} |v(x)|^p \right\}^{1/p}.$$

Let us suppose that v is defined on ω_h (or on $\overline{\omega}_h$ and equal to zero on γ_h). The discrete Fourier transform $\tilde{v}$ of v is given by

$$\tilde{v}_k = \sum_{x \in \omega_h} h^2 v(x) \sin(k_1 \cdot x_1) \sin(k_2 \cdot x_2), \qquad k = (k_1, k_2), x = (x_1, x_2).$$

The inverse discrete Fourier transform of v is defined by

$$\hat{v}(x) = \left(\frac{2}{\pi} \right)^2 \sum_{k \in K_h} v_k \sin(k_1 \cdot x_1) \sin(k_2 \cdot x_2),$$

with $K_h = \{ k = (k_1, k_2) \in \mathbf{N} \times \mathbf{N} \colon 0 < k_j h < \pi, j = 1,2 \}$.

A function v, defined on ω_h (or on $\overline{\omega}_h$ and equal to zero on γ_h), is said to belong to $W^{r,p}(\omega_h)$, $-\infty < r < \infty$, $1 < p < \infty$, if there exists a function $V \in L^p(\omega_h)$ such that

$$v(x) = (I_{r,h} V)(x) = \widehat{\left(1 + |k|^2\right)^{-r/2} \tilde{V}_k}.$$

By definition, we set

$$\|v\|_{r,p,h} = \|V\|_{p,h}.$$

We now turn to a generalization of a discrete interpolation inequality established by Mokin [6].

LEMMA 2. *Let α and β be two nonnegative real numbers such that $\alpha < \beta$. If $v \in W^{\beta,p}(\omega_h)$, $1 < p < \infty$, there exists a positive constant C, independent of h, such that for any real number r, $\alpha \leqslant r \leqslant \beta$,*

$$\|v\|_{r,p,h} \leqslant C \|v\|_{\alpha,p,h}^{1-\mu} \|v\|_{\beta,p,h}^{\mu},$$

with $\mu = (r - \alpha)/(\beta - \alpha)$.

Proof. Since the statement is true for $\alpha = 0$ [6], we shall assume that $\alpha > 0$. Let $w = I_{-\alpha,h} v$. It follows that

$$\|w\|_{\beta-\alpha,p,h} = \|v\|_{\beta,p,h} \quad \text{and} \quad \|w\|_{r-\alpha,p,h} = \|v\|_{r,p,h}.$$

Moreover,

$$\|w\|_{r-\alpha,p,h} \leqslant C\|w\|_{p,h}^{1-\mu}\|w\|_{\beta-\alpha,p,h}^{\mu}$$

and the desired inequality follows immediately.

Consider the finite difference scheme

$$-\Delta_h z = \overline{\nabla}_1\nabla_1\eta_1 + \overline{\nabla}_2\nabla_2\eta_2, \qquad x \in \omega_h, \tag{3}$$

$$z(x) = 0, \qquad x \in \gamma_h, \tag{4}$$

with η_j defined on $\omega_h \cup \gamma_h^j$ and equal to zero on γ_h^j, $j = 1, 2$.

An easy argument based on the discrete multiplicator techniques [6] shows that

$$\|z\|_{2,p,h} \leqslant C\left(\|\overline{\nabla}_1\nabla_1\eta_1\|_{p,h} + \|\overline{\nabla}_2\nabla_2\eta_2\|_{p,h}\right), \tag{5}$$

$$\|z\|_{1,p,h} \leqslant C\left(\|[\nabla_1\eta_1\|_{p,h} + \|[\nabla_2\eta_2\|_{p,h}\right), \tag{6}$$

$$\|z\|_{p,h} \leqslant C\left(\|\eta_1\|_{p,h} + \|\eta_2\|_{p,h}\right), \tag{7}$$

with a positive constant C, independent of z and h.

Let us suppose that the solution u of boundary value problem (1), (2) belongs to $W^{s,p}(\Omega)$, $s > 2/p$, $1 < p < \infty$. By Sobolev's imbedding theorem [8], u is continuous on $\Omega \cup \partial\Omega$ and

$$\left(T_{20}\frac{\partial^2 u}{\partial x_1^2}\right)(x) = \overline{\nabla}_1\nabla_1 u(x), \qquad x \in \omega_h,$$

$$\left(T_{02}\frac{\partial^2 u}{\partial x_2^2}\right)(x) = \overline{\nabla}_2\nabla_2 u(x), \qquad x \in \omega_h.$$

Therefore,

$$\overline{\nabla}_1\nabla_1 T_{02}u + \overline{\nabla}_2\nabla_2 T_{20}u = -(T_{22}f)(x), \qquad x \in \omega_h, \tag{8}$$

$$u(x) = 0, \qquad x \in \gamma_h. \tag{9}$$

Thus, if the solution of boundary value problem (1), (2) belongs to $W^{s,p}(\Omega)$, $s > 2/p$, $1 < p < \infty$, we may associate with (1), (2) the finite difference scheme

$$\Delta_h v = -(T_{22}f)(x), \qquad x \in \omega_h, \tag{10}$$

$$v(x) = 0, \qquad x \in \gamma_h. \tag{11}$$

Error estimates will be given in Section 5.

Let us turn to the case when u, the solution of boundary value problem (1), (2), belongs to $W^{s,p}(\Omega)$, $0 \leqslant s < 1 + 1/p$, $1 < p < \infty$. Define

$$\dot{W}^{s,p}(\Omega) = \begin{cases} W^{s,p}(\Omega), & 0 \leqslant s \leqslant 1/p, 1 < p < \infty, \\ \{w\colon w \in W^{s,p}(\Omega), w = 0 \text{ on } \partial\Omega\}, & 1/p < s < 1 + 1/p, 1 < p < \infty \end{cases}$$

and observe that $u \in \dot{W}^{s,p}(\Omega)$, $0 \leqslant s < 1 + 1/p$, $1 < p < \infty$. Let $\Omega^* = (-\pi, 2\pi) \times (-\pi, 2\pi)$. The extension of u by 0 outside Ω is a continuous mapping of $\dot{W}^{s,p}(\Omega)$ into $W^{s,p}(\Omega^*)$, $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$, $1 < p < \infty$ [5], [8]. Hence,

$$u \mapsto u^* = \text{odd extension of } u$$

is a continuous mapping of $\dot{W}^{s,p}(\Omega)$ into $W^{s,p}(\Omega^*)$, $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$, $1 < p < \infty$.

It is obvious that

$$(T_{11}u^*)(x) = 0, \qquad x \in \gamma_h,$$
$$(T_{31}u^*)(x) = 0, \qquad x \in \gamma_h^1,$$
$$(T_{13}u^*)(x) = 0, \qquad x \in \gamma_h^2.$$

We introduce $f^* \in \mathcal{D}'(\Omega^*)$ by

$$f^* = -\Delta u^*.$$

Whence,

$$\overline{\nabla}_1\nabla_1 T_{13}u^* + \overline{\nabla}_2\nabla_2 T_{31}u^* = -(T_{33}f^*)(x), \qquad x \in \omega_h, \tag{12}$$

$$(T_{11}u^*)(x) = 0, \qquad x \in \gamma_h. \tag{13}$$

Therefore, if the solution of (1), (2) belongs to $W^{s,p}(\Omega)$, $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$, $1 < p < \infty$, we may associate with (1), (2) the finite difference scheme

$$\Delta_h v = -(T_{33}f^*)(x), \qquad x \in \omega_h, \tag{14}$$

$$v(x) = 0, \qquad x \in \gamma_h. \tag{15}$$

Error estimates will be given in Section 5.

5. Convergence of Finite Difference Schemes.

THEOREM 1. *Let u be the solution of boundary value problem* (1), (2), *v the solution of the discrete problem* (10), (11) *and $k \in \{0,1,2\}$. If $u \in W^{s,p}(\Omega)$, $2/p < s \leqslant k + 2$, $1 < p < \infty$, the following error estimate holds*

$$\|u - v\|_{k,p,h} \leqslant Ch^{s-k}|u|_{s,p,\Omega},$$

with a positive constant C independent of h. Moreover, if $s \geqslant k$ then finite difference scheme (10), (11) *converges in the discrete norm $\|\cdot\|_{k,p,h}$.*

Proof. (a) Let us suppose first that $k = 2$. By (8)–(11) it follows that the function $z = v - u$ is defined on $\overline{\omega}_h$ and satisfies (3), (4) with $\eta_1 = u - T_{02}u$ and $\eta_2 = u - T_{20}u$. Now, η_j is defined on the grid $\omega_h \cup \gamma_h^j$ and equal to zero on γ_h^j, $j = 1, 2$. Thanks to inequality (5), it suffices to estimate $\|\overline{\nabla}_j\nabla_j\eta_j\|_{p,h}$, $j = 1, 2$. We introduce the squares

$$E(i_1, i_2) = \{x = (x_1, x_2) \in \mathbf{R}^2 \colon (i_j - 1)h \leqslant x_j \leqslant (i_j + 1)h, j = 1, 2\},$$
$$E = \{t = (t_1, t_2) \in \mathbf{R}^2 \colon -1 \leqslant t_j \leqslant 1, j = 1, 2\},$$

and the affine mapping $x = (x_1, x_2) \in E(i_1, i_2) \mapsto t = (t_1, t_2) \in E$ with $x_j = i_j h + t_j h$, $j = 1, 2$. Let us set $\tilde{u}(t) = u(x(t))$. Then

$$\begin{aligned}\overline{\nabla}_1\nabla_1\eta_1(i_1 h, i_2 h) &= \frac{u(i_1 h + h, i_2 h) - 2u(i_1 h, i_2 h) + u(i_1 h - h, i_2 h)}{h^2} \\ &\quad - \int_{-1}^{1} \theta_2(s)\frac{u(i_1 h + h, i_2 h + sh) - 2u(i_1 h, i_2 h + sh) + u(i_1 h - h, i_2 h + sh)}{h^2}\,ds \\ &= \frac{1}{h^2}\left\{\tilde{u}(1,0) - 2\tilde{u}(0,0) + \tilde{u}(-1,0) - \int_{-1}^{1} \theta_2(s)(\tilde{u}(1,s) - 2\tilde{u}(0,s) + \tilde{u}(-1,s))\,ds\right\}.\end{aligned}$$

Furthermore, $\overline{\nabla}_1 \nabla_1 \eta_1(i_1 h, i_2 h)$ is a bounded linear functional on $W^{s,p}(E)$, $s > 2/p$, with a kernel $\supset \mathbf{P}^3(E)$. By Lemma 1,

$$|\overline{\nabla}_1 \nabla_1 \eta_1(i_1 h, i_2 h)| \leqslant \frac{c}{h^2} |\tilde{u}|_{s,p,E}$$

for $2/p < s \leqslant 4$. Thus,

$$|\overline{\nabla}_1 \nabla_1 \eta_1(i_1 h, i_2 h)| \leqslant \frac{c}{h^2} \cdot h^{s-2/p} |u|_{s,p,E(i_1,i_2)}$$

for $2/p < s \leqslant 4$. Finally,

$$\|\overline{\nabla}_1 \nabla_1 \eta_1\|_{p,h} \leqslant c h^{s-2} |u|_{s,p,\Omega}, \qquad 2/p < s \leqslant 4.$$

Likewise,

$$\|\overline{\nabla}_2 \nabla_2 \eta_2\|_{p,h} \leqslant c h^{s-2} |u|_{s,p,\Omega}, \qquad 2/p < s \leqslant 4,$$

and that completes the proof for $k = 2$.

(b) Let $k = 1$. By (6) it suffices to estimate $|[\nabla_j \eta_j\|_{p,h}$, $j = 1,2$. In the same manner as in (a) we conclude that

$$\nabla_1 \eta_1(i_1 h, i_2 h) = \frac{1}{h}\left\{ \tilde{u}(1,0) - \tilde{u}(0,0) - \int_{-1}^{1} \theta_2(s)(\tilde{u}(1,s) - \tilde{u}(0,s))\, ds \right\}$$

is a bounded linear functional on $W^{s,p}(E)$, $s > 2/p$, with a kernel $\supset \mathbf{P}^2(E)$. Therefore,

$$|[\nabla_1 \eta_1\|_{p,h} \leqslant c h^{s-1} |u|_{s,p,\Omega}, \qquad 2/p < s \leqslant 3,$$

and, similarly,

$$|[\nabla_2 \eta_2\|_{p,h} \leqslant c h^{s-1} |u|_{s,p,\Omega}, \qquad 2/p < s \leqslant 3.$$

That completes the proof for $k = 1$.

(c) Finally, let $k = 0$. Let us estimate $\|\eta_j\|_{p,h}$, $j = 1,2$. Since

$$\eta_1(i_1 h, i_2 h) = \tilde{u}(0,0) - \int_{-1}^{1} \theta_2(s) \tilde{u}(0,s)\, ds$$

is a bounded linear functional on $W^{s,p}(E)$, $s > 2/p$, with a kernel $\supset \mathbf{P}^1(E)$, thanks to Lemma 1,

$$\|\eta_1\|_{p,h} \leqslant c h^{s} |u|_{s,p,\Omega}, \qquad 2/p < s \leqslant 2,$$

and, similarly,

$$\|\eta_2\|_{p,h} \leqslant c h^{s} |u|_{s,p,\Omega}, \qquad 2/p < s \leqslant 2.$$

By (7) we obtain the desired error estimate.

Lemma 2 enables us to derive scales of error estimates.

THEOREM 2. *Let u be the solution of boundary value problem* (1), (2) *and v the solution of discrete problem* (10), (11). *If $u \in W^{s,p}(\Omega)$, $2/p < s \leqslant 2$ and $0 \leqslant r \leqslant 2$, or $2/p < s \leqslant 3$ and $1 \leqslant r \leqslant 2$, $1 < p < \infty$, the following error estimate holds*

$$\|u - v\|_{r,p,h} \leqslant C h^{s-r} |u|_{s,p,\Omega},$$

with a positive constant C independent of h. Moreover, if $s \geqslant r$ then finite difference scheme (10), (11) *converges in the discrete norm $\|\cdot\|_{r,p,h}$.*

Proof. Let us suppose that u belongs to $W^{s,p}(\Omega)$, $2/p < s \leqslant 2$, $1 < p < \infty$ and $0 \leqslant r \leqslant 2$. We apply Lemma 2 with $\alpha = 0$, $\beta = 2$ and Theorem 1 to derive the desired estimate.

Likewise, if u belongs to $W^{s,p}(\Omega)$, $2/p < s \leqslant 3$, $1 < p < \infty$ and $1 \leqslant r \leqslant 2$, set $\alpha = 1$ and $\beta = 2$ to conclude the estimate.

Let us turn to finite difference scheme (14), (15).

THEOREM 3. *Let u be the solution of boundary value problem* (1), (2), *v the solution of discrete problem* (14), (15), *and $k \in \{0,1\}$. If $u \in W^{s,p}(\Omega)$, $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$, $1 < p < \infty$, then*

$$\|T_{11}u - v\|_{k,p,h} \leqslant Ch^{s-k}|u|_{s,p,\Omega},$$

with a positive constant C independent of h. Moreover, if $s \geqslant k$, then finite difference scheme (14), (15) *converges in the discrete norm $\|\cdot\|_{k,p,h}$.*

Proof. (a) Let $k = 1$. By (12)–(15) it follows that the function $z = v - T_{11}u^*$, defined on $\bar{\omega}_h$, satisfies (3), (4) with $\eta_1 = T_{11}u^* - T_{31}u^*$ and $\eta_2 = T_{11}u^* - T_{13}u^*$. The function η_j is defined on $\omega_h \cup \gamma_h^j$ and equal to zero on γ_h^j, $j = 1,2$. We define

$$E(i_1, i_2) = \left\{x = (x_1, x_2) \in \mathbf{R}^2 \colon \left(i_j - \tfrac{3}{2}\right)h \leqslant x_j \leqslant \left(i_j + \tfrac{3}{2}\right)h, j = 1,2\right\},$$

$$E = \left\{t = (t_1, t_2) \in \mathbf{R}^2 \colon -\tfrac{3}{2} \leqslant t_j \leqslant \tfrac{3}{2}, j = 1,2\right\}$$

and the affine mapping $x = (x_1, x_2) \in E(i_1, i_2) \mapsto t = (t_1, t_2) \in E$ with $x_j = i_jh + t_jh$, $j = 1,2$. Let $\tilde{u}(t) = u^*(x(t))$. Then

$$\nabla_1\eta_1(i_1h, i_2h) = \frac{1}{h}\left\{\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left(\tilde{u}(s_1 + 1, s_2) - \tilde{u}(s_1, s_2)\right) ds_1\, ds_2 - \int_{-1/2}^{1/2}\int_{-3/2}^{3/2}\theta_3(s_1)\left(\tilde{u}(s_1 + 1, s_2) - \tilde{u}(s_1, s_2)\right) ds_1\, ds_2\right\}$$

is a bounded linear functional on $W^{s,p}(E)$, $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$ with a kernel $\supset \mathbf{P}^2(E)$. By Lemma 1,

$$|\nabla_1\eta_1(i_1h, i_2h)| \leqslant \frac{c}{h}|\tilde{u}|_{s,p,E}, \qquad 0 \leqslant s < 1 + 1/p, s \neq 1/p.$$

Thus,

$$|\nabla_1\eta_1(i_1h, i_2h)| \leqslant \frac{c}{h}h^{s-2/p}|u^*|_{s,p,E(i_1,i_2)} \leqslant \frac{c}{h}h^{s-2/p}|u|_{s,p,E(i_1,i_2)\cap\Omega}, \qquad 0 \leqslant s < 1 + 1/p, s \neq 1/p,$$

since the extension $u \in \dot{W}^{s,p}(\Omega) \mapsto u^* \in W^{s,p}(\Omega^*)$ is continuous for $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$, $1 < p < \infty$. Finally,

$$\|[\nabla_1\eta_1\|_{p,h} \leqslant Ch^{s-1}|u|_{s,p,\Omega}, \qquad 0 \leqslant s < 1 + 1/p, s \neq 1/p.$$

Similarly,

$$\|[\nabla_2\eta_2\|_{p,h} \leqslant Ch^{s-1}|u|_{s,p,\Omega}, \qquad 0 \leqslant s < 1 + 1/p, s \neq 1/p.$$

By (6) we conclude the desired estimate.

(b) For $k = 0$ we apply the same technique as in (a).

We make use of Theorem 3 and Lemma 2 with $\alpha = 0$, $\beta = 1$ to prove the following theorem.

THEOREM 4. *Let u be the solution of boundary value problem* (1), (2) *and v the solution of discrete problem* (14), (15). *If $u \in W^{s,p}(\Omega)$, $0 \leqslant s < 1 + 1/p$, $s \neq 1/p$, $1 < p < \infty$ and $0 \leqslant r \leqslant 1$, then*

$$\|T_{11}u - v\|_{r,p,h} \leqslant Ch^{s-r}|u|_{s,p,\Omega},$$

with a positive constant C independent of h. Moreover, if $s \geqslant r$, then finite difference scheme (14), (15) *converges in the discrete norm $\|\cdot\|_{r,p,h}$.*

Remark 7. If $1/p < s < 1 + 1/p$, then $f \in W^{s-2,p}(\Omega)$ implies $u \in \dot{W}^{s,p}(\Omega)$ [8]. Furthermore, if $0 < s < 1/p$ then $f \in \Xi^{s-2,p}(\Omega)$ implies $u \in W^{s,p}(\Omega)$ [5], [8]. Since $\Delta u^* = (\Delta u)^*$ for $u \in \dot{W}^{s,p}(\Omega)$, $0 < s < 1 + 1/p$, $T_{33}f^*$ may be calculated from f. Therefore finite difference scheme (14), (15) is applicable.

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